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The Duffing Oscillator in the Low-Friction Limit: Theory and Analog Simulation

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By a joint use of theory and analog simulation the low-friction regime of the Duffing oscillator is explored. In the weak-temperature case it is shown that the low-friction regime, in turn, must be divided in two well-distinct subregimes. In the former one, characterized by the friction γ ranging from $2\omega_0$ (ω_0 is the harmonic frequency) up to a lower bound γ_T, the method of statistical linearization applies and a continued fraction procedure (CFP) generated by the Zwanzig–Mori projection techniques is shown to provide correct information for both the renormalized frequency of the oscillator and the corresponding line shape. The latter subregime, characterized by the friction γ ranging from $\gamma = 0$ to $\gamma = \gamma_T$, is fraught with the complete breakdown of the statistical linearization method. The CFP is shown to provide an incorrect picture of the line shape while suggesting a novel mean field approximation which is then proven analytically via an alternative method of calculation. This method consists of expressing the system in a form reminiscent of the model of Kubo's stochastic oscillator. By using this alternative approach we are in a position to account for the residual linewidth which is shown by the analog experiment to survive for $\gamma \to 0$.

KEY WORDS: Nonlinear stochastic oscillator; Zwanzing–Mori projection procedure; multiplicative stochastic oscillator; renormalized frequency; statistical linearization.

1. INTRODUCTION

The Duffing oscillator is the simplest model for nonlinear systems subject to random interactions. This explains why in the recent past a large num-

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ber of articles have appeared[1-11] discussing approximate solutions to this anharmonic stochastic oscillator. This model is described by

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -(\omega_0^2 x + \beta x^3) - \gamma v + f(t) \end{aligned} \tag{1.1}$$

This means that a Brownian particle with space coordinate x and velocity v moves under the influence of an external anharmonic potential, a friction term, and a stochastic force $f(t)$, which is assumed to be Gaussian and δ correlated, i.e.,

$$\langle f(t) \rangle = 0, \qquad \langle f(0)\, f(t) \rangle = 2D\delta(t) \equiv 2\gamma k_B T \delta(t) \tag{1.2}$$

The corresponding Fokker–Planck equation reads

$$\begin{aligned} \frac{\partial}{\partial t} \rho(x, v; t) &= \mathscr{L} \rho(x, v; t) \\ &\equiv \left[-v \frac{\partial}{\partial x} + (\omega_0 x + \beta x^3) \frac{\partial}{\partial v} + \gamma \left(\frac{\partial}{\partial v} v + k_B T \frac{\partial^2}{\partial v^2} \right) \right] \rho(x, v; t) \end{aligned} \tag{1.3}$$

In the present paper this Fokker–Planck equation will be written in such a form as to make the relevant anharmonicity strength α to appear. This parameter is defined by

$$\alpha \equiv \frac{3}{4} \frac{\beta}{\omega_0^3} k_B T \tag{1.4}$$

We do not pretend to provide an exhaustive and a fair picture of the wide work concerning the challenging field of the nonlinear stochastic processes: Certainly we are prevented from doing that by our own cultural background and personal bias. Nevertheless to point out that the present investigation on the low-friction regime led us to discover a completely unexpected physical behavior, we must compare our study with a well-defined group of previous papers. These are divided into three categories.

1.1. Continued Fraction Expansion of the Spectral Density

The paper of Bixon and Zwanzig[3] is an outstanding example of this procedure. They evaluated up to nine frequency moments of the spectral density, i.e., used four terms of the continued fraction expansion. A further example of a continued fraction expansion is the paper of Matsuo,[8] who evaluated up to 56 frequency moments, i.e., about 28 steps of the continued fraction expansion. Although largely enough to get a convergence in the high-friction regime, a truncation of the continued fraction at about the same order as Matsuo is proven by the results of this paper to lead, in the low-friction case, to completely unreliable results. The continued fraction procedure (CFP) used in this paper is virtually an improvement of that developed by Bixon and Zwanzig.[3] This procedure will be shortly reviewed in Section 2, and more details on it can be found in Refs. 12–14. By using this computer algorithm we are in a position to evaluate up to 40 steps of the continued fraction expansion. The major result arrived at in this paper via application of the CFP is that the renormalizated oscillator frequency changes from $\Omega_{\text{eff}} = \omega_0 + 2\alpha$ into $\Omega_{\text{eff}} = \omega_0 + \alpha$ with decreasing the friction parameter γ from $\gamma \gg \alpha$ to $\gamma \ll \alpha$. It is also shown that the low-friction regime ($\gamma < \alpha$) is characterized by a residual linewidth which cannot be accounted for by using the CFP (unless a novel very efficient resummation technique is found).

1.2. Mean Field Approximation

A major purpose of the wide research work on the Duffing oscillator is to find a linear transport equation which leads to the same average values as the actual nonlinear microscopic equation of Langevin kind on which the proper description of this system should rely. According to Bixon and Zwanzig[3] this linear equation should be restricted to small initial deviations from equilibrium. We shall show that this statement must be supplemented by the requirement that

$$\alpha \ll \gamma \tag{1.5}$$

A simple illustration of the linearization procedure is as follows. Let us consider the equilibrium correlation function

$$\Phi_x(t) \equiv \langle x(0)\, x(t) \rangle / \langle x^2 \rangle \tag{1.6}$$

from, e.q., (1.3) we obtain

$$\frac{d}{dt} \Phi_x(t) = \langle x(0)\, v(t) \rangle / \langle x^2 \rangle \tag{1.6a}$$

$$\frac{d}{dt} \frac{\langle x(0)\, v(t) \rangle}{\langle x^2 \rangle} = -\omega_0^2 \Phi_x(t) - \gamma \frac{\langle x(0)\, v(t) \rangle}{\langle x^2 \rangle} - \beta \frac{\langle x(0)\, x^3(t) \rangle}{\langle x^2 \rangle} \tag{1.6b}$$

We observe that the last term on the right-hand side of Eq. (1.6b) generates an infinite hierarchy of equations. The simplest way of truncating this hierarchy is to make the mean field assumption:

$$\langle x(0)\, x^3(t)\rangle = 3\langle x^2\rangle\langle x(0)\, x(t)\rangle \tag{1.7}$$

The factor of 3 comes out from the fact that in the weak-temperature region

$$[\langle x(0)\, x^3(t)\rangle_{\text{eq}}]_{t=0} = \langle x^4\rangle \simeq 3\langle x^2\rangle^2 \tag{1.7a}$$

When only the first two steps of the corresponding expansion are used, the CFP[12–14] produces the same result. It is therefore correct to state, as Matsuo did,[8] that the continued fraction expansion method includes the method of the statistical linearization. We see indeed that Eq. (1.7a) leads to the same renormalized frequency as the method of the "statistical linearization",[6,7,11] From Eqs. (1.5) to (1.7a) we get the renormalized frequency defined by

$$\Omega_{\text{eff}} = (\omega_0^2 + 3\beta k_B T/\omega_0^2)^{1/2} \simeq \omega_0 + 2\alpha \tag{1.8}$$

which is precisely the same result as that provided by the statistical linearization. Further remarks on this issue may be found in Ref. 2.

The major result of the present paper on this issue is that the mean-field approximation leading to Eq. (1.8) fails completely in the low-friction regime characterized by

$$\gamma \lesssim \alpha \tag{1.9}$$

By using a completely analytical theory it will be shown indeed that a novel mean-field approximation must be applied leading to (for $\gamma \ll \alpha$)

$$\Omega_{\text{eff}} = \omega_0 + \alpha \tag{1.10}$$

1.3. Analog Simulation

The work of Morton and Corrsin[1] and that of Bulsara *et al.*[2] are the unique analog simulations of the Duffing oscillator known to us. These interesting works left unexplored the extremely weak friction region of Eq. (1.9). To explore this regime via analog simulation required significant technical improvements, which will be illustrated in Section 5. It is worth remarking that the role played by analog simulation in this paper is determinant. The clear indications of the experimental results served the main purpose of questioning this seemingly undoubted prediction of the CFP (see the discussion in Fig. 4): In the extremely low-friction regime the distribution of the frequencies higher than ω_0 is "quantized" rather than continuous. This does not agree with the existence provided via the analog experiment, of a residual line width which survives for $\gamma \to 0$. Thus the theory developed in Section 4 has been largely suggested by the analog simulation. Note that, although it is relatively straightforward to establish an analytical result for the deterministic case $\gamma = 0$ (no stochastic force is present), it is not immediately clear whether or not this is really the limit for $\gamma \to 0$ of the actual stochastic system.

The outline of the present paper is as follows. Section 2 is devoted to illustrating the results of the CFP. Section 3 provides an intuitive picture for the low-friction regime characterized by $\gamma \ll \alpha$. Section 4 is devoted to showing that this regime requires a novel mean-field approximation which naturally stems from a representation of the system as a multiplicative stochastic oscillator. The "experimental results" obtained via analog simulation are described in Section 5, which is also devoted to a comparison between "experiment" and the theoretical predictions.

2. THE CONTINUED FRACTION PROCEDURE

To make the present paper as self-contained as possible, we shall shortly review the CFP of Refs. 12–14. This method relies on the fact that the evaluation of a stationary correlation function implies that the system reaches a well-defined steady state. The Fokker–Planck equation of Eq. (1.3) is proven to have the equilibrium solution of canonical form

$$\rho_{\mathrm{eq}}(x, v) \propto \exp\left[-\left(\frac{\omega_0^2 x^2}{2} + \frac{\beta x^4}{4} + \frac{v^2}{2}\right)\Big/ k_B T\right] \tag{2.1}$$

If we define the scalar product between two observables $A(x, v)$ and $B(x, v)$ as

$$\langle A \mid B \rangle \equiv \int A^*(x, v)\, B(x, v)\, \rho_{\mathrm{eq}}(x, v)\, dx dv \tag{2.2}$$

the correlation function

$$\Phi_a(t) = \langle a(0)\, a(t) \rangle / \langle a^2 \rangle \tag{2.3}$$

where $a(x, v)$ is a variable depending on x and v can then be given the "quantum mechanical" form

$$\Phi_a(t) \equiv \langle f_0 \mid f_0(t) \rangle / \langle f_0 \mid f_0 \rangle \tag{2.4}$$

where

$$|f_0\rangle \equiv a(x, v) \tag{2.5}$$

$|f_0\rangle$ is thought of as the first of a chain of states (the Mori basis set[13]), which is the best expansion basis set of the operator of Eq. (1.3) when aiming at determining the correlation function $\Phi_a(t)$. This is indeed a generalization of the well-known Mori theory[15] to the case of non-Hermitian operators.[12,13]

The chain of the Mori states is built up as follows:

$$|f_1\rangle = -\lambda_0 \,|\, f_0\rangle + \mathscr{L} \,|\, f_0\rangle \tag{2.6}$$

$$\langle \tilde{f}_1| = -\langle \tilde{f}_0|\, \lambda_0 + \langle f_0|\, \mathscr{L}^+ \tag{2.6a}$$

$$|f_i\rangle = -\lambda_{i-1}\, |f_{i-1}\rangle - \Delta_{i-1}^2\, |f_{i-2}\rangle + \mathscr{L}\, |f_{i-1}\rangle, \qquad i = 2,... \tag{2.7}$$

$$\langle \tilde{f}_i| = -\lambda_{i-1}\langle \tilde{f}_{i-1}| - \Delta_{i-1}^2\langle \tilde{f}_{i-2}| + \langle \tilde{f}_{i-1}|\, \mathscr{L}^+, \qquad i = 2,... \tag{2.7'}$$

$$\Delta_i^2 \equiv -\langle \tilde{f}_i \,|\, f_i\rangle / \langle \tilde{f}_{i-1} \,|\, f_{i-1}\rangle, \qquad i = 1,... \tag{2.8}$$

$$\lambda_i \equiv \langle \tilde{f}_i|\, \mathscr{L}\, |f_i\rangle / \langle \tilde{f}_i \,|\, f_i\rangle, \qquad i = 0, 1,... \tag{2.9}$$

Note that $\mathscr{L}^+$ is the Hermitian conjugate of $\mathscr{L}$ with respect to the scalar product defined by Eq. (2.2). The evaluation of the states $|f_i\rangle$ and $\langle \tilde{f}_i|$ via an iterative approach leads to the direct determination of the fundamental parameters λ_i and Δ_i^2 [Eqs. (2.8) and (2.9)]. The iterative evaluation of the states $|f_i\rangle$, in turn, is possible owing to the fact that $\mathscr{L}\,|f_i\rangle$ can be given the form

$$\mathscr{L}\, |f_i\rangle = \sum_{nm} c_{nm}^{(i)} x^n v^m \exp\left\{ -\left[V(x) + \frac{v^2}{2} \right] \Big/ k_B T \right\} \tag{2.10}$$

This affords a complete solution of the problem under discussion, because the spectrum of the variable a is given by

$$\mathrm{I}_a(\omega) \equiv \mathrm{Re} \int_0^\infty e^{-i\omega t} \Phi_a(t)\, dt \equiv \mathrm{Re}\, \hat{\Phi}_a(i\omega) \tag{2.11}$$

and $\hat{\Phi}_a(i\omega)$, in turn, can be expressed[12,13] in the continued fraction form

$$\hat{\Phi}_a(i\omega) = \cfrac{1}{i\omega + \lambda_0 + \cfrac{\Delta_1^2}{i\omega + \lambda_1 + \cfrac{\Delta_2^2}{i\omega + \lambda_2 + \cdots}}} \tag{2.12}$$

By using a computer algorithm we can determine about 40 Mori states. The method can be improved via exact resummation over the infinite terms of the expansion. Some special cases lead to an exact resummation of the continued fraction. These are the following.

2.1. Kubo's Stochastic Oscillator[16]

The well-known model of the multiplicative stochastic oscillator of Kubo reads

$$\dot{\alpha}_{\pm}(t) = i\eta(t)\,\alpha_{\pm}(t) \tag{2.13}$$

where $\eta(t)$ is a Gaussian stochastic variable defined by

$$\langle \eta(t) \rangle = \Omega \tag{2.14}$$

$$\langle \Delta\eta(0)\,\Delta\eta(t) \rangle = \Delta^2 \exp\{-\gamma t\} \tag{2.14'}$$

$$\Delta\eta \equiv \eta - \langle \eta \rangle = \eta - \Omega \tag{2.15}$$

By following Kubo[16] we get

$$\int_0^{\infty} \langle \alpha_-(0)\,\alpha_+(t) \rangle\, e^{-zt}\, dt = \cfrac{\langle \alpha_- \alpha_+ \rangle}{z - i\Omega + \cfrac{\Delta^2}{z - i\Omega + \gamma + \cfrac{2\Delta^2}{z - i\Omega + 2\gamma + \cfrac{3}{\ddots}}}} \tag{2.16}$$

where

$$\langle \alpha_-(0)\,\alpha_+(t) \rangle = \langle \alpha_- \alpha_+ \rangle \exp\left[-\frac{\Delta^2}{\gamma^2}(e^{-\gamma t} - 1 + \gamma t) + i\Omega t \right] \tag{2.17}$$

On the other hand, for $\gamma \to 0$, $\langle \alpha_- \alpha_+(t) \rangle_{\text{eq}}$ of Eq. (2.17) becomes

$$\Phi_\alpha(t) = \langle \alpha_- \alpha_+(t) \rangle = \langle \alpha_- \alpha_+ \rangle\, e^{-\Delta^2 t^2/2} e^{i\Omega t} \tag{2.18}$$

The Laplace transform of which is

$$\hat{\Phi}_\alpha(z) = \frac{1}{\Delta}\, e^{(i\Omega - z)^2/2\Delta^2} \int_{-(i\Omega - z)/\Delta}^{\infty - (i\Omega - z)/\Delta} e^{-x^2/2}\, dx \tag{2.19}$$

We therefore conclude that for $\gamma \to 0$ and $\Omega = 0$ Eq. (2.19) expresses the result of summing the continued fraction of Eq. (2.16) to infinite order.

2.2. A One-Dimensional Chain of Harmonic Oscillators[17]

Let us consider the system described by

$$\begin{aligned}
\dot{x}_0 &= v_0 \\
\dot{v}_0 &= -\kappa(x_0 - x_1) \\
\dot{x}_1 &= v_1 \\
\dot{v}_1 &= \kappa(x_0 - x_1) - \kappa(x_1 - x_2) \\
\dot{x}_2 &= v_2 \\
\dot{v}_2 &= \kappa(x_1 - x_2) - \kappa(x_2 - x_3) \\
&\vdots
\end{aligned} \tag{2.20}$$

It is well known[18] that

$$\Phi_{v0}(t) \equiv \langle v_0 v_0(t) \rangle / \langle v_0^2 \rangle = J_0(2\sqrt{\kappa}\, t) \tag{2.21}$$

the Laplace transform of which is

$$\hat{\Phi}_{v0}(z) = \frac{1}{(z^2 + 4\kappa)^{1/2}} \tag{2.22}$$

On the other hand, from Eqs. (2.20) it turns out[17] that

$$\hat{\Phi}_{v0}(z) = \cfrac{1}{z + \cfrac{2\Delta^2}{z + \cfrac{\Delta^2}{z + \cfrac{\Delta^2}{z + \cdots}}}} \tag{2.23}$$

Equation (2.22) is therefore the exact summation of the infinite continued fraction of Eq. (2.23).

2.3. The Lorents model of turbulence[19]

Grossmann and Sonnenborg-Schmick[19] have recently studied the chaotic properties of the Lorentz model through representative dynamical-correlation functions. By using the Mori–Zwanzig projector formalism they expressed these correlation function in a continued fraction form. They could evaluate up to seven Mori states, which were proven to be far from being enough to explain the major features of the Lorentz spectra.

However these authors could assess that the Lorentz turbulent regime is characterized by the following power law:

$$\Delta_i^2 = i^\nu \Delta^2 \tag{2.24}$$

In their case ν turned out to be $\nu = 2$. We shall refer to this type of continued fraction as the Lorentz model of turbulence.[19] Note that by summing this continued fraction to the infinite order Grossmann and Sonnenberg-Schmick obtained a broad spectrum that satifactorily agrees with the experimental one, whereas truncation at the sixth stage resulted in only two sharp transitions.[19]

We note that[21]

$$\int_0^\infty \frac{b \sin \text{h}(x)}{\sin \text{h}(bx)} \exp(-zx) = \cfrac{1}{z + \cfrac{\Delta_1^2}{z + \cfrac{\Delta_2^2}{z + \Delta_3^2 \cdots}}} \tag{2.25}$$

where

$$\Delta_i^2 = \frac{(i^2 b^2 - 1)}{4i^2 - 1} i^2 \tag{2.26}$$

When $i \gg 1$ this simulates fairly well the power law of the Lorentz model of turbulence. Since in the present paper summation to infinite order will be made after actually calculating about 40 Mori states, Eq. (2.25) will approximate fairly well the power law of Eq. (2.24).

The summation to infinite order will be carried out as follows. Let us assume that

$$\hat{\Phi} \equiv \hat{\Phi}_0(z) = \cfrac{1}{z - \lambda_0 + \cfrac{\Delta_1^2}{z - \lambda_1 + \cfrac{\Delta_2^2}{\ddots}}} \tag{2.27}$$

is a function, which can also be given by a fairly simple analytical expression. Note that

$$\hat{\Phi}_i(z) = \frac{\Delta_i^2}{z - \lambda_i + \hat{\Phi}_{i+1}(z)} \tag{2.28}$$

Then through iterative application of the inverse relation

$$\hat{\Phi}_{i+1}(z) = \frac{\Delta_i^2}{\hat{\Phi}_i(z)} - z + \lambda_i \tag{2.29}$$

we can express a generic $\hat{\Phi}_n(z)$ in terms of the parameters $\lambda_0, \lambda_1, ..., \lambda_{n-1}, \Delta_1^2, \Delta_2^2, ..., \Delta_{n-1}^2$ and the analytical expression $\hat{\Phi}(z)$. This analytical expression will be provided by one of the afore mentioned cases.

Figure 1 illustrates a significant result of the CFP procedure applied to evaluating the correlation function $\Phi_x(t)$ of the Duffing oscillator. This has been the subject of a preliminary short report.[22] We see that the normalized frequency changes from $\Omega_{\text{eff}} = \omega_0 + 2\alpha$ at $\gamma \gg \alpha$ into $\Omega_{\text{eff}} = \omega_0 + \alpha$ at $\gamma < \alpha$. This means that the regime $\gamma < \alpha$ is characterized by a complete failure of the usual renormalization technique.

Figure 2 illustrates the spectra $\text{Re}\,\hat{\Phi}_x(i\omega)$ at different values of γ/α. Special attention is deserved by the spectrum in the regime $\gamma \ll \alpha$, which is characterized by two sharp peaks. This is in marked contrast with the result of the analog simulation of Section 5, which proves the existence of a residual linewidth the size of which is about α. To shed ligtht into this disagreement we evaluated via the CFP the behavior of the parameters Δ_i^2 as a function of the parameter γ. We noted (see Fig. 3a) that when $\gamma \gg \alpha$ these parameters undergo a weak fluctuation at low values of the index i. As a

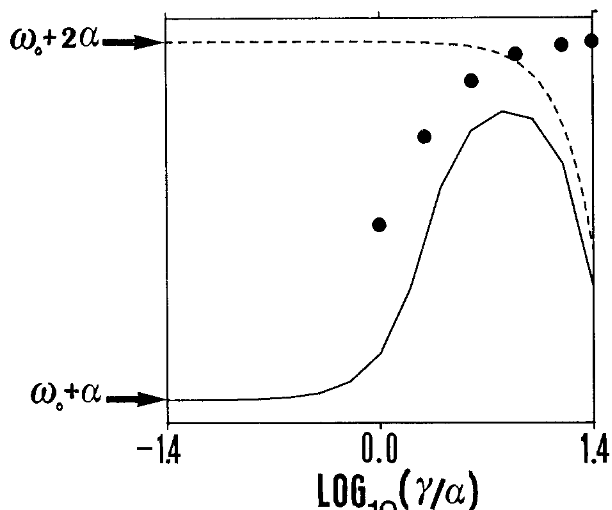

Fig. 1. The maximum of $\text{Re}\,\hat{\Phi}_x(i\omega)$ [the real part of the Laplace transform of $\Phi_x(t)$ of Eq. (1.6)] as a function of the friction parameter γ. The solid line illustrates the result obtained by using the CFP of this section, whereas the line denoted by large dots indicates the result provided by Eq. (4.27). The discrepancy between the two results at large values of γ mainly depends on the fact that there the conditions for the rotating-wave approximation behind Eq. (4.6) are not fulfilled. The dashed line denotes $(\Omega_{\text{eff}}^2 - \gamma^{2/4})^{1/2}$, where Ω_{eff} is provided by the statistical linearization of Eq. (1.8), and shows therefore that at $\gamma \gg \alpha$ this mean-field method gives correct results. The parameter α used is: $\alpha = 0.0075$.

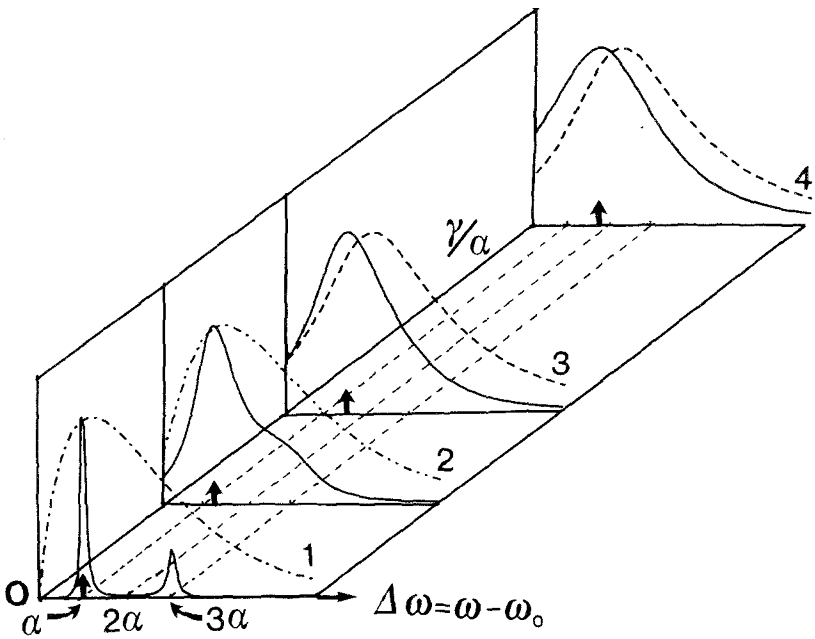

Fig. 2. The real part of the Laplace transform of $\Phi_x(t)$, Re $\hat{\Phi}_x(i\omega)$, at several values of the friction γ. (1) $\gamma/\alpha = 0$, (2) $\gamma/\alpha = 1/2$, (3) $\gamma/\alpha = 1$, (4) $\gamma/\alpha = 2$. The solid line denotes the result provided the CFP with 35 Mori states and standard truncation (i.e., without additional summation at infinite order). The curves -·-·- at $\gamma = 0$ and -···- at $\gamma = \alpha/2$ denotes the result provided by Eq. (4.20). The curves --- at $\gamma = \alpha$ and $\gamma = 2\alpha$ denote the result provided by Eq. (4.27). The arrows denote the position of the maximum of the solid lines. The abscissas concern $\Delta\omega = \omega - \omega_0$, the origin of which is pointed by 0. The scale of ordinates, concerning Re $\hat{\Phi}_x(i\omega)$, is expressed in arbitrary units. The parameter α used is $\alpha = 0.0075$.

certain threshold i_T is reached these parameters start exhibiting a much more fluctuating dependence on the index i. This allows us to truncate the Mori chain at the (i_T-)th order without affecting the resulting spectrum with any significant error. On the contrary, the regime $\gamma \lesssim \alpha$ is characterized by a more regular behavior which makes it difficult tc truncate the Mori chain at any order. Figure 3b clearly shows that the regular behavior of the region $\gamma \lesssim \alpha$ may be expressed via the power law of Eq. (2.24) with $\nu \simeq 2.5$.

Figure 4 illustrates the results of applying different kinds of summation at infinite order. It is shown that all the different methods virtually lead to the same result: a major peak at $\omega \simeq \omega_0 + \alpha$ and a satellite one at about $\omega \simeq \omega_0 + 3\alpha$. This would suggest the following interpretation picture: Weak as it is, the stochastic force $f(t)$ cf the Eq. (1.1) is able to establish (via the fluctuation-dissipation process) the temperature T. This means that from time to time the Brownian particle is obliged to leave the bottom of the

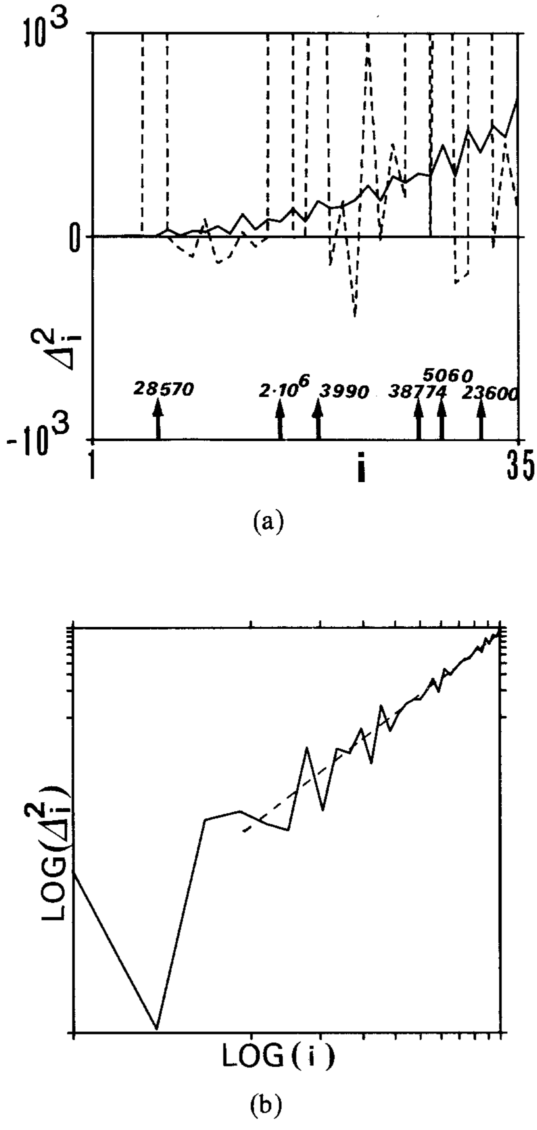

Fig. 3. The behavior of the parameters Δ_i^2 as a function of i. In both cases the value of the parameter α used is $\alpha = 0.075$. (a) Linear plot pointing out the change from a fluctuating behavior to a regular one when the region $\gamma \sim \alpha$ is reached. Dashed line, $\gamma = 10\alpha$; solid line, $\gamma = 6\alpha$. All the cases with $\gamma \leqslant 6\alpha$ virtually coincide with the solid line. The arrows denote the values of the corresponding parameters Δ_i^2 out of scale. (b) The behavior of the parameters Δ_i^2 as a function of i illustrated via a bilogarithmic plot. The full line refers to the case $\gamma = 0$. The dashed line indicates $\Delta_i^2 = i^\nu$ with $\nu = 2.5$.

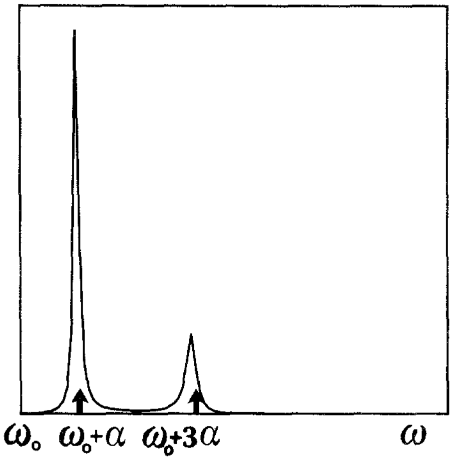

Fig. 4. Re $\hat{\Phi}_x(i\omega)$ as a function of ω. The curve has been obtained by evaluating 35 states of the set of Eq. (2.6) to Eq. (2.7′), and making a subsequent summation at infinite order were based on the model (2). If this summation were made by using either the model (1) or (3) this would provide results indistinguishable in the scale of this figure from that based on the model (2). Note that virtually the same result would be obtained by summation over 440 states (the first 40 states being rigorously determined via Eqs. (2.6) to (2.7′) and the remaining ones via an extrapolation method based on Eq. (2.24) with $\nu = 2.5$) and subsequent summation at infinite order with any one of the models (1) to (3). Ordinates refer to Re $\hat{\Phi}_x(i\omega)$ expressed in arbitrary units. $\alpha = 0.075$, $\gamma = 0$.

well and explore regions of higher potential energy. Then, in consequence of the anharmonic contribution to the potential V the Brownian particle "feels" effective frequencies larger than ω_0. The results of the CFP would imply, on the other hand, that the distribution of these effective frequencies is "quantized" rather than continuous, as it should be within the framework of the classical mechanics, and the constant of quantization would be precisely the anharmonic strength α [Eq. (1.4)]. However, both analog experiment and theory (see next section) show that these sharp transitions, if they exist, are embodied within a broad spectrum. This leads us to believe that for this broad spectrum to be reproduced by the CFP it may be necessary to make a summation at infinite order closer to the power law $\nu \simeq 2.5$ which characterizes the extremely low-friction regime. Note that the turbulent regime of Lorentz is characterized by the power law: $\nu = 2$.[19] This seems to suggest that the residual linewidth of size α at $\gamma = 0$ may be the effect of a transition from the standard stochastic regime, depending on the influence of infinite irrelevant freedom degrees, to the deterministic chaos. This is a sound hypothesis because the system cf

Eq. (1.1) can be regarded[18] as being a collection of infinite systems such as

$$\begin{aligned}
\dot{x} &= v \\
\dot{v} &= -(\omega_0^2 x + \beta x^2) - \kappa'(x - y_0) \\
\dot{y}_0 &= w_0 \\
\dot{w}_0 &= -\kappa(y_0 - x) - \kappa(y_0 - y_1) \\
\dot{y}_1 &= w_1 \\
\dot{w}_1 &= \kappa(y_0 - y_1) - \kappa(y_1 - y_2) \\
&\vdots
\end{aligned} \tag{2.30}$$

with a canonical equilibrium distribution. Each system of the collection is characterized by a well-defined energy E which is a constant of the motion. If it happened that at a certain critical energy E_c a transition to chaos due to the anharmonic term of the Duffing oscillator takes place,[2] then beyond this threshold the spectrum of the variables x and v would be determined by the deterministic noise rather than depending on the standard fluctuation-dissipation process caused by the interaction between the Brownian particle and the "thermal bath" particles with coordinates $y_0, y_1, \ldots, y_i, \ldots$ [simulated in Eq. (1.1) by the friction term $-\gamma v(t)$ and the stochastic force $f(t)$]. Of course the study of the transition to chaos of the system of Eq. (2.30) affords evident technical difficulties. We focused therefore our attention on

$$\begin{aligned}
\dot{x} &= v \\
\dot{v} &= -(\omega_0^2 x + \beta x^3) - \Omega^2(x - y_0) \\
\dot{y} &= w_0 \\
\dot{w} &= -\Omega^2(y_0 - x).
\end{aligned} \tag{2.31}$$

If the residual linewidth exibited at $\gamma < \alpha$ depended on this transition to chaos, one would expect that the system of Eq. (2.31) is characterized by a critical energy given by

$$E_c \sim \frac{4}{3} \frac{\omega_0^3 \Omega}{\beta} \tag{2.32}$$

This prevision is obtained from the mere definition of α of Eq. (1.4) by replacing $k_B T$ with E_c and γ with Ω (which in a sense has the same meaning as γ, i.e., a rate of energy exchange between the system of interest and its "thermal bath").

[2] For a few papers on the transition to chaos in the Duffing oscillator, see Refs. 23a to 23c. Closely related to this subject are also the papers of Refs. 23d and 23e.

We explored values of the energy E up to $E \simeq 10^2 E_c$ without detecting any deviation from the standard deterministic behavior. This means that the power law with $\nu = 2.5$ must be regarded as being nothing but a mere sign of strong non-Markoffian behavior preventing us from using the usual truncations of the Mori chain. The similarity between this and the Lorentz turbulent regime[19] probably only implies that the spectrum of chaotic systems with a few freedom degrees shares some common features with those non-Markoffian systems the chaotic behavior of which depends on the influence of infinite "irrelevant" freedom degrees.

In the next two sections we shall illustrate the actual physical reason behind the residual linewidth characterizing the regime $\gamma < \alpha$.

3. A DETERMINISTIC MODEL FOR THE EXTREMELY LOW-FRICTION REGIME

For $\gamma \to 0$ the trajectories of the Brownian particles are virtually deterministic. An extremely weak stochastic force from time to time produces transitions from a deterministic orbit to another one. When dealing with a sample of Brownian particles, these must be regarded as being distributed according to a canonical law, each one undergoing a completely deterministic motion characterized by a well-defined energy E. Owing to the nonlinear character of the potential

$$V(x) = \tfrac{1}{2}\omega_0^2 x^2 + \tfrac{1}{4}\beta x^4 \tag{3.1}$$

a finite temperature distribution makes frequencies higher than the harmonic one, ω_0, appear. A particle which undergoes oscillations of amplitude A is characterized by the fundamental frequency

$$\omega^2 = \omega_0^2 + \tfrac{3}{4}\beta A^2, \qquad \beta A^2 \ll \omega_0^2 \tag{3.2}$$

This expression is approximated and corresponds to the first step of a frequency renormalization procedure[24] widely used to solve the equation of motion in nonlinear systems. In the case under discussion, either by means of numerical integration or using the Jacobian elliptical functions, it would be straightforward to reach a better approximation to this frequency. In general this depends on the amplitude A of the oscillation and therefore on the energy of the particle:

$$\omega = \omega(E) \tag{3.3}$$

$$E \equiv \tfrac{1}{2}\omega_0^2 A^2 + \tfrac{1}{4}\beta A^4 \tag{3.4}$$

To find the explicit form of the autocorrelation function we will make the assumption that the motion of the particle be represented only by its first harmonic component:

$$x(t) = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t \tag{3.5}$$

In order to evaluate $\langle x(0)\, x(t) \rangle_{eq}$ we must average over all the possible initial conditions taking into account the weight given by

$$\rho(E, x) \propto 2e^{-E/k_B T} \frac{1}{\{2[E - V(x)]\}^{1/2}} \tag{3.6}$$

which is obtained from the canonical distribution function by making a transformation from the variables x, v to the new ones x and E. We obtain therefore

$$\langle x(0)\, x(t) \rangle \propto 2 \int_0^\infty dE \int_{-A}^{A} dx_0 \exp\left(-\frac{E}{k_B T}\right) \frac{x_0^2 \cos \omega(E)\, dt}{\{2[E - V(x_0)]\}^{1/2}} \tag{3.7}$$

the Laplace transform of which leads us to expect

$$\mathbf{I}_x(\omega) \propto 2e^{-E/k_B T} \frac{dE}{d\omega} \int_{-A}^{A} \frac{x_0^2\, dx_0}{\{2[E - V(x_0)]\}^{1/2}} \tag{3.8}$$

where ω and E are related to one another through Eq. (3.3), or, in an approximated form, Eq. (3.2). The assumption that α [defined by Eq. (1.4)] is small allows us to evaluate the integral appearing in Eq. (3.8) and the normalization factor involved by Eq. (3.6), thereby leading us to

$$\mathbf{I}_x(\omega) = \frac{\pi}{2} e^{-E/k_B T} \frac{dE}{d\omega} E \tag{3.9}$$

The factor of $\pi/2$ comes from the definition of Eq. (2.11). In order to derive Eq. (3.8) we made only the assumption of neglecting the third- and higher-order harmonics in the solution of the equation of motion. This is certainly a valid one in the limit of small α. The prediction of Eq. (3.8) supplemented by Eq. (3.2) will be shown to agree very well with the experimental results of Section 5 (see Fig. 7).

As to the energy diffusion this approach can be extended to the case of finite friction parameter γ by adopting the Stratonovich method[25] widely applied by the La Jolla group.[26] In the next section, however, we shall follow a different approach relying on the Fokker–Planck equation of (1.3) and the rotating-wave approximation. It is proven that in the limit case of small α and vanishingly small γ, the result of this deterministic model, Eq. (3.9), coincides with Eq. (4.20).

4. A NOVEL MEAN FIELD APPROXIMATION

Let us write the Fokker–Planck aquation of Eq. (1.3) in terms of the new variables

$$\alpha_{\pm} \equiv v \pm i\omega_0 x \tag{4.1}$$

We then obtain

$$\begin{aligned}\frac{\partial}{\partial t}\rho(\alpha_+, \alpha_-; t) = \Bigg\{ & -i\omega_0\left(\frac{\partial}{\partial \alpha_+}\alpha_+ - \frac{\partial}{\partial \alpha_-}\alpha_-\right) \\ & + \frac{i\beta}{8\omega_0^3}\left(\frac{\partial}{\partial \alpha_+} + \frac{\partial}{\partial \alpha_-}\right)(\alpha_+ - \alpha_-)^3 \\ & + \frac{\gamma}{2}\left[\left(\frac{\partial}{\partial \alpha_+} + \frac{\partial}{\partial \alpha_-}\right)(\alpha_+ + \alpha_-)\right. \\ & \left. + 2\kappa_B T\left(\frac{\partial}{\partial \alpha_+} + \frac{\partial}{\partial \alpha_-}\right)\left(\frac{\partial}{\partial \alpha_+} + \frac{\partial}{\partial \alpha_-}\right)\right]\Bigg\}\rho(\alpha_+, \alpha_-; t)\end{aligned} \tag{4.2}$$

Let us consider the first term on the r.h.s. of Eq. (4.2) as being the unperturbed part $\mathscr{L}_0$ of the operator $\mathscr{L}$. When written in the corresponding interaction picture, Eq. (4.2) reads

$$\frac{\partial}{\partial t}\tilde{\rho}(\alpha_+, \alpha_-; t) = \mathscr{L}_1(t)\,\tilde{\rho}(\alpha_+, \alpha_-; t) \tag{4.3}$$

where

$$\mathscr{L}_1(t) \equiv e^{-\mathscr{L}_0 t}\mathscr{L}_1 e^{\mathscr{L}_0 t} \tag{4.4}$$

and

$$\begin{aligned}\mathscr{L}_1 \equiv & \frac{i\beta}{8\omega_0^3}\left(\frac{\partial}{\partial \alpha_+} + \frac{\partial}{\partial \alpha_-}\right)(\alpha_+ - \alpha_-)^3 + \frac{\gamma}{2}\left[\left(\frac{\partial}{\partial \alpha_+} + \frac{\partial}{\partial \alpha_-}\right)(\alpha_+ + \alpha_-)\right. \\ & \left. + 2k_B T\left(\frac{\partial}{\partial \alpha_+} + \frac{\partial}{\partial \alpha_-}\right)\left(\frac{\partial}{\partial \alpha_+} + \frac{\partial}{\partial \alpha_-}\right)\right]\end{aligned} \tag{4.5}$$

Let us assume

$$\omega_0 \gg \Delta\Gamma \tag{4.6}$$

where $\Delta\Gamma$ is the linewidth of the spectrum of the variables of interest, x and v. We are then allowed to replace $\mathscr{L}_1(t)$ with its time average over the fast time oscillations with frequencies $\pm 2i\omega_0$. We thus obtain

$$\frac{\partial}{\partial t}\tilde{\rho}(\alpha_+, \alpha_-; t) = \overline{\mathscr{L}_1(t)}\,\tilde{\rho}(\alpha_+, \alpha_-; t) \tag{4.7}$$

where

$$\begin{aligned}\mathscr{L}_1 \equiv & -\frac{3i\beta}{8\omega_0^3}\left(\frac{\partial}{\partial\alpha_+}\alpha_+^2\alpha_- - \frac{\partial}{\partial\alpha_-}\alpha_-^2\alpha_+\right)\\ & + \frac{\gamma}{2}\left(\frac{\partial}{\partial\alpha_+}\alpha_+ + \frac{\partial}{\partial\alpha_-}\alpha_- + 4k_BT\frac{\partial}{\partial\alpha_+}\frac{\partial}{\partial\alpha_-}\right)\end{aligned} \tag{4.8}$$

Note that in this rotating frame of reference the system appears to be characterized by the equilibrium distribution

$$\tilde{\rho}(\alpha_+, \alpha_-) \propto \exp\left(-\frac{\alpha_+\alpha_-}{2k_BT}\right) \tag{4.9}$$

This implies an assumption of weakly anharmonic interaction. We shall show indeed that for $\gamma \to 0$ the linewidth is dominated by the anharmonic strength α [Eq. (1.4)]. In such a case the condition of Eq. (4.6) results in

$$\alpha \ll \omega_0 \tag{4.10}$$

i.e., precisely an assumption of weakly anharmonic interaction. When the condition of Eq. (4.10) is fulfilled we are allowed to use this reference system, where the variable energy reads

$$E = \alpha_+\alpha_-/2 \tag{4.11}$$

It is now convenient to make a new change of variables. We replace α_+ and α_- with α_+ and E. This is equivalent to replacing Eqs. (4.7) and (4.8) with

$$\begin{aligned}&\frac{\partial}{\partial t}\tilde{\rho}(\alpha_+, E; t)\\ &\quad \equiv \left[-i\frac{3}{4}\frac{\beta}{\omega_0^3}\left(\frac{\partial}{\partial\alpha_+}\alpha_+\right)E + \gamma k_BT\left(\frac{\partial}{\partial\alpha_+}\alpha_+\right)\frac{\partial}{\partial E}\right.\\ &\quad\quad \left. + \frac{\gamma}{2}\frac{\partial}{\partial\alpha_+}\alpha_+ + \gamma\left(\frac{\partial}{\partial E}E + k_BT\frac{\partial}{\partial E}E\frac{\partial}{\partial E}\right)\right]\tilde{\rho}(\alpha_+, E; t)\end{aligned} \tag{4.12}$$

Note that the last change of variables is especially suitable when exploring the region where

$$\gamma \ll \alpha \tag{4.13}$$

We see indeed from Eq. (4.12) that the frequency of oscillation of the variable α_+ is precisely α (note that $\langle E \rangle = k_B T$), whereas the variable E decays with the rate $1/\gamma$. When the condition of Eq. (4.13) is fulfilled, the variable α_+ must be considered as being much faster than E. Since the derivative $\partial/\partial\alpha_+$ always appears on the left side, contraction over the variable α_+ cancels all the terms on the right-handside of Eq. (4.12) but the last one. We thus obtain the energy diffusion equation

$$\frac{\partial}{\partial t}\sigma(E;t) = \gamma\left(\frac{\partial}{\partial E}E + k_B T\frac{\partial}{\partial E}E\frac{\partial}{\partial E}\right)\sigma(E;t) \tag{4.14}$$

which is characterized by the equilibrium distribution

$$\sigma_{\text{eq}}(E) \propto \exp(-E/k_B T) \tag{4.15}$$

We note also that the motion of E does not depend on the variable α_+. The time evolution of α_+, on the contrary, is deeply dependent on E. From Eq. (4.12) we indeed obtain

$$\dot{\alpha}_+(t) = i\frac{3}{4}\frac{\beta E}{\omega_0^3}\alpha_+(t) - \frac{\gamma}{2}\alpha_+(t) \tag{4.16}$$

From Eq. (4.16) we get

$$\alpha_-(0)\,\alpha_+(t) = 2\exp\left[\left(i\frac{3}{4}\beta E - \frac{\gamma}{2}\right)t\right]E \tag{4.17}$$

To determine the correlation function $\langle \alpha_-(0)\,\alpha_+(0)\rangle$ we must make an average over the equilibrium distribution of Eq. (4.15). This leads to

$$\langle \alpha_-\alpha_+(t)\rangle = 2kTe^{-(\gamma/2)t}\frac{1}{(1 - i\alpha t)^2} \tag{4.18}$$

It is also evident that

$$\langle \alpha_+(0)\,\alpha_+(t)\rangle = 0 \tag{4.18'}$$

We must now come back to the laboratory system. It can be shown that [we use Eq. (4.18′)]

$$\langle x(0)\, x(t)\rangle = \frac{Re}{2\omega_0^2} \langle \alpha_-(0)\, \alpha_+(t)\rangle\, e^{i\omega_0 t} \tag{4.19}$$

This leads to the spectrum

$$\mathbf{I}_x(\omega) = \mathrm{Re} \int_0^\infty \frac{k_B T}{\omega_0^2} \frac{[(1-\alpha^2 t^2)\cos \omega_0 t - 2t\alpha \sin \omega_0 t]}{(1+\alpha^2 t^2)^2} e^{-(\gamma/2)t} e^{-i\omega t}\, dt \tag{4.20}$$

Equation (4.18) shows that the correlation function $\langle \alpha_-(0)\, \alpha_+(t)\rangle$ changes sign at $t = 1/\alpha$ thereby implying the presence of the frequency α into $\mathbf{I}_x(\omega)$. Note that this oscillation frequency could be arrived at by replacing the factor of 3 of Eq. (1.7) with 3/2. This would be disconcerting without the analysis of the present section, which leads indeed to a completely novel mean field approximation. In other words, in the weak friction regime it is still possible to replace the real anharmonic oscillator with an effective "linear" one provided that Kubo's picture of stochastic oscillator(16) is used, i.e.,

$$\frac{d}{dt} \alpha_+(t) = -i\eta(t)\, \alpha_+(t) \tag{4.21}$$

where the stochastic frequency $\eta(t)$ is characterized by

$$\langle \eta(t)\rangle = \alpha \tag{4.22}$$

The variance of η can be evaluated by remarking that η can be identified with the variable energy E. Thus we have

$$\langle (\eta - \alpha)^2 \rangle = \alpha^2 \tag{4.23}$$

Kubo's equations(16) can then be suitably adapted to deal with this case, thereby providing $(\Delta\omega \equiv \omega - \omega_0)$

$$\mathbf{I}(\omega) = \exp\left[-\frac{(\Delta\omega - \alpha)^2}{\alpha^2} \right] \tag{4.24}$$

which is certainly worse than Eq. (4.20), since Kubo's formula(16) rests on the assumption that η is a Gaussian variable and loses therefore the asymmetric character of Eq. (4.20), character which is corroborated by the experimental results of the next section. However, this is to remark that the residual linewidth for $\gamma \to 0$ is no doubt the same kind of phenomenon as that behind Kubo's stochastic oscillator in the highly non-Markovian regime.

Equation (4.12) can also be used to provide results which are not confined to the case $\gamma \ll \alpha$. If we define

$$\Phi_n(t) \equiv \langle \alpha_- E^n \alpha_+(t) \rangle \tag{4.25}$$

from Eq. (4.12) we obtain

$$\frac{d}{dt}\Phi_n = i\alpha\Phi_n - \gamma\left(n+\frac{1}{2}\right)\Phi_{n+1} + \gamma n(n+1)\,\Phi_{n-1} \tag{4.26}$$

By using the iterative procedure illustrated in Ref. 29 we get

$$\hat{\Phi}_0(z) = \hat{F}_0(z)(\langle\varepsilon\rangle + \hat{F}_1(z)\,a_0^1(\langle\varepsilon^2\rangle + \hat{F}_2(z)\,a_1^2(\langle\varepsilon^3\rangle + \hat{F}_3(z)\,a_2^3(\langle\varepsilon^4\rangle\cdots))) \tag{4.27}$$

where

$$\hat{F}_0(z) = \cfrac{1}{z-\lambda_0 - \cfrac{a_0^1 a_1^0}{z-\lambda_1 - \cfrac{a_1^2 a_2^2}{z-\lambda_2}}} \tag{4.27$'$}$$

$$\hat{F}_1(z) = \cfrac{1}{z-\lambda_1 - \cfrac{a_1^2 a_2^1}{z-\lambda_2 - a^3 a^3}} \tag{4.27$''$}$$

and so on. Note that $a_n^{n-1} \equiv \gamma n(n+1)$, $a_n^{n+1} \equiv i\alpha$, $\lambda_n \equiv -\gamma(n+1/2)$ and $\varepsilon \equiv E/kT$. The last definition leads to $\langle\varepsilon^n\rangle = n\langle\varepsilon^{n-1}\rangle$, $\langle\varepsilon\rangle = 1$.

This expression can be applied to studying the region $\gamma \gg \alpha$ where it is proven to predict the maximum of $\mathbf{I}_x(\omega)$ at $\omega = \omega_0 + 2\alpha$. Of course the plateau at $\gamma \gg \alpha$ predicted by Eq. (4.27) (see Fig. 1) depends on the fact that this equation relies on the rotating-wave approximation and therefore should be compared with the result provided by the CEP in the limit $\omega_0 \to \infty$.

Before closing this section, we would like to stress a remarkable result provided by the theory here developed. The regime where statistical linearization applies is mainly dominated by the influence of the standard fluctuation-dissipation processes related with the terms $-\gamma v$ and $f(t)$ of Eq. (1.1). On the contrary at $\gamma \sim \alpha$ the system enters a new regime with a line shape very close to that of the purely deterministic case $\gamma = 0$ (see Figs. 1 and 2). As a major result of this paper, it is therefore shown that the deterministic character of the system is dominant throughout the whole interval $0 \leqslant \gamma \sim \alpha$.

5. ANALOG SIMULATION AND COMPARISON BETWEEN THEORY AND EXPERIMENT

The experimental apparatus is basically a development of that used in Ref. 27. Furthermore, more details can be found in a separate paper.[28] For the sake of clarity we shall limit ourselves to illustrating the "experimental" apparatus via the scheme of Fig. 5.

The Duffing equation

$$\ddot{x} = -\gamma \dot{x} - \omega_0^2 x - \beta x^3 \tag{5.1}$$

was simulated using two integrators coupled with two multipliers. The voltage V_0 (Fig. 5) was kept fixed so as to get the linear term $\omega_0^2 x$ at the end of the second multiplier. The damping γ was changed by means of a resistance of feedback in the first integrator.

To determine the spectrum of the variable x a Gaussian white noise was applied to the input of the analog device. The x output was sent to a computer so as to evaluate the power spectrum.

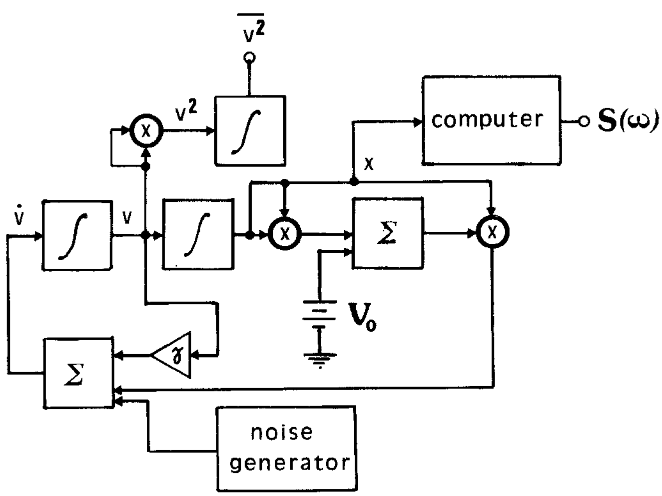

Fig. 5. Scheme of the experimental apparatus.

The extremly low-friction regime that we are exploring in this paper involves some technical difficulties:

(a) The low values of γ are obtained from high values of the feedback resistance of the electric device.

(b) In this case the electric circuit is particularly sensible to the internal noise because the amplitude of the noise applied was reduced so as to keep $\langle x^2 \rangle_{eq}$ fixed.

(c) The largely inertial character of the system obliges us to make averages over extremely long intervals of time. This in turn enhances the drift resulting from the thermal variations of the electric components.

The limitations (a) and (b) were bypassed by choosing the parameter α sufficiently large, though still small compared to ω_0 (In the case explored in this section we used $\alpha = 0.067\omega_0$). The thermal drift (c) was avoided by making the measurements after waiting the time necessary for the electric components to reach their stationary values. Of course, the cautions we used are not sufficient for the results of the measurements to be totally reproducible, and little fluctuations still haunt the spectrum.

In Fig. 6 we show this spectrum at several values of γ. The transition

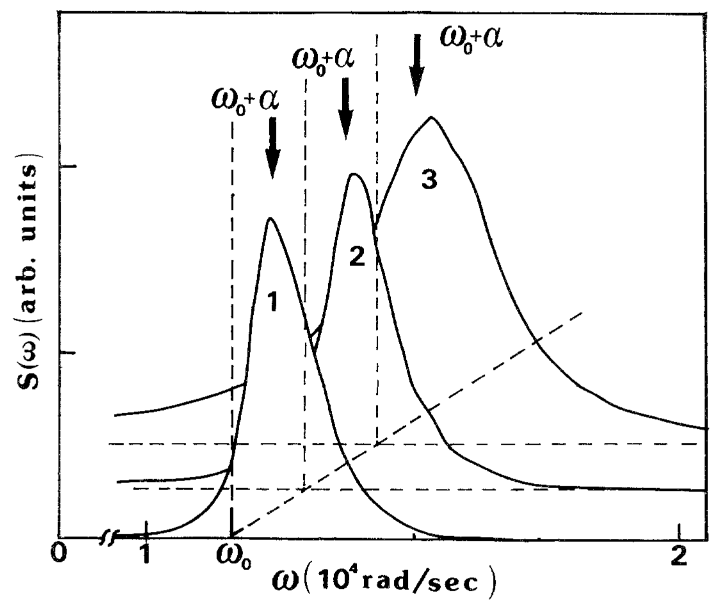

Fig. 6. The free relaxation spectrum of the variable x at different values of the friction γ. The curves 1, 2, and 3 concern $\gamma = 0.011\omega_0$, $\gamma = 0.04\omega_0$, $\gamma = 0.182\omega_0$, respectively. The arrows denote the positions of the frequency $\omega_0 + \alpha$, where $\omega_0 = 11\,600$ rad/sec and $\alpha = 0.067\omega_0$.

from the regime where Eq. (1.8) holds to that where we must apply Eq. (1.10) is clearly shown by this figure.

In Fig. 7 we compare the theoretical predictions on $\gamma \ll \alpha$ with the result of analog simulation. The good agreement between theory and experiment corrohorates our statement that (at about $\gamma \sim \alpha$) a transition takes place from the region where statistical linearization holds to another regime which requires the novel mean field approximation illustrated in Section 5 (which is proven by Fig. 7 to predict the position of the peak with satisfactory precision).

We must also conclude that the appearance of sharp peaks at $\gamma < \alpha$ is an artifact of the CFP which should be still more marked when less effective computer algorithms are used. Thus analog simulation played a decisive role in leading us to reach the correct view, according to which the sharp satellite peak appearing at $\omega = \omega_0 + 3\alpha$ is the sign of an asymmetric broadband, peaked at $\omega = \omega_0 + \alpha$. Note that the distance between the major peak and the satellite one is 2α, which is precisely of the same order of magnitude as the residual broadening.

A previous case of investigation on the extremely low-friction limit is that of Risken.[29] His work concerns the case of a cosine potential.

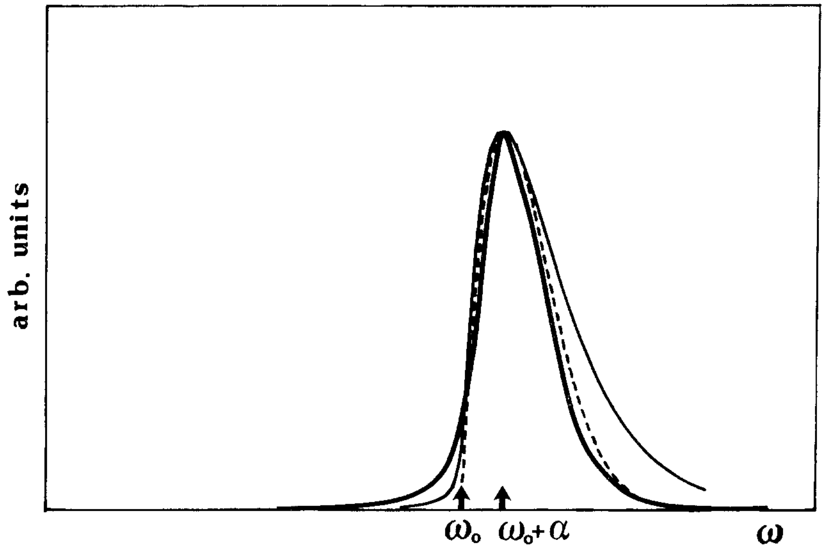

Fig. 7. The spectrum of the free relaxation of the variable x in the extremely low-friction regime. The solid lines describe the spectrum of Eq. (4.20). The dashed line describes the deterministic spectrum of Eq. (3.8) supplemented by Eq. (3.2). The result of analog simulation is given by the large full line. The parameters used are $\alpha = 0.067\omega_0$, $\gamma = 0.0108\omega_0$, and $\omega_0 = 11\ 600$ rad/sec. The arrows denote the frequencies ω_0 and $\omega_0 + \alpha$.

However, in the low-temperature limit this is not distinguishable from the model of Eq. (1.1) with $\beta < 0$. When this is taken into due account, his analytical result is proven to coincide with that of Eq. (4.20), thereby completely corroborating our point of view. Ref. 31 shows furthermore how to extend the approach of the present paper to the case of Josephson junction potential.

6. CONCLUDING REMARKS

The major results of this paper are the following.

(i) The CFP of Ref. 14 which is basically founded on the Zwanzing–Mori projection method and therefore closely related to the Bixon-Zwanzing method[3] is virtually reliable over the whole friction interval, if our attention is focused on the maximum of the spectrum. This paper shows that the CFP allows us to discover the transition from the region where (in the low-temperature approximation) this maximum is placed at $\omega = \omega_0 + 2\alpha$, to the new one where it appears at $\omega = \omega_0 + \alpha$. This also establishes a lower bound to the techniques of statistical linearization, placed at about $\gamma \sim \alpha$. In this region, left unexplored by the previous investigations based on the continued fraction method, i.e., Refs. 3 and 8, the CFP provides wrong information on the detailed shape of the spectrum, although the position of its center of gravity remains correct. A problem left not completely solved by this paper is whether or not a proper summation at infinite order exists which allows the CFP to provide a more reliable linewidth.

(ii) The transition from $(\Delta\omega)_{\max} = 2\alpha$ to $(\Delta\omega)_{\max} = \alpha$ is completely corroborated by the result of analog simulation (see Fig. 7).

(iii) A new theory is established, based on the rotating-wave approximation, which is correct (as far as both the line shape and maximum are concerned) throughout the whole friction range. This theory shows that an interpretation in terms of an equivalent "linear" oscillator is still possible, provided that this oscillator is given the form of a Kubo stochastic oscillator.[16] The discrepancies between this theory and the results of the analog simulation must principally be ascribed to the fact that for technical reasons the analog experiment cannot explore those regions where the errors coming from the rotating-wave approximation are really negligible (α cannot be made much smaller than ω_0). To corroborate this statement, let us consider Fig. 7. The discrepancy between the result of Eq. (3.8) and that of Eq. (4.20) depends, in part, on the fact that the former one applies to the case $\gamma = 0$, whereas the latter concerns $\gamma \simeq 0.1\alpha$. This remark applies especially to the low-frequency side. However, γ is not large enough as to account for the disagreement between Eq. (3.8) and Eq. (4.20) in the high-frequency side, which has to be ascribed therefore to the rotating-wave approximation. Thus by inspection of Fig. 7 we see that the rotating-wave approximation is largely responsible for the discrepancy between the theory of Section 4 and the experimental results.

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